The One Loop Effective Super-Potential and Non-Holomorphicity

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Abstract

We calculate the Kählerian and the lowest order non-Kählerian contributions to the one loop effective superpotential using super-Feynman graphs in the massless Wess-Zumino model, the massive Wess-Zumino model and N = 1 U(1) gauge theory. We also calculate the Kählerian term in Yang-Mills theory for a general gauge group. Using this latter result we find the one loop Kählerian contribution for $N = 2$ Yang-Mills theory in terms of $N = 1$ superfields and we show that it can only come from non-holomorphic contributions to the $N = 2$ effective potential.

The effective potential, Γ, is the effective action with its external momenta, p_i, set to zero [1]. For a supersymmetric theory we can calculate the one loop contributions of lowest order in the covariant derivatives using the Feynman rules in $N = 1$ superspace. These are explained along with our conventions in Ref. [3] which also contains extensive references to the original literature. For a generic $N = 1$ supersymmetric theory with chiral superfields, Φ, and gauge fields, V, the effective potential can be written as

$$\int d^8z\, K(\bar{\Phi}, \Phi, V) + F_1(\bar{\Phi}, \Phi, V)\, D^A \Phi D_A \Phi \bar{D}^{\dot{B}} \bar{\Phi} \bar{D}_{\dot{B}} \bar{\Phi}$$

$$+ F_2(\bar{\Phi}, \Phi, V)\, D^A \bar{D}^{\dot{B}} V\, D_A \bar{D}_{\dot{B}} V + \ldots + O(D^4, \bar{D}^4) \tag{1}$$

The first term, K, in Eq. (1) is referred to as the Kählerian term. The first order non-Kählerian terms contain exactly two D_A and two $\bar{D}_{\dot{B}}$ derivatives and the . . . indicate that we have not shown the terms where covariant derivatives act on both V and either Φ or $\bar{\Phi}$.

To illustrate our techniques we first calculate the Kählerian term and the lowest order non-Kählerian term, F_1, for the massless Wess-Zumino model, which has no V fields, and then show how to extend our results to the massive Wess-Zumino model. The Kählerian terms were found by a different method in Ref. [2]. In this method the superfields ϕ and $\bar{\phi}$ are expanded about their background values and the superpropagator is found for the quantum perturbations. The one loop effective potential is expressed in terms of this superpropagator which is in turn expressed in terms of a heat kernel by the Schwinger-de-Witt representation. This results in a system of

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coupled differential equations which can be solved to give an expression for the effective potential. Enforcing the restriction $D_A\Phi = 0$ gives the Kählerian term. The authors go on to give the form of the second term in the expansion, F_1, but they do not calculate its coefficient explicitly. We shall show how this can be done using supergraphs.

Following this we apply similar methods to N = 1 U(1) gauge theory. In this case there is also an F_2 term and $W_A = g\bar{D}^2 D_A V$, so we can get the $F_2 W^A W_A$ term from this. In Yang-Mills theory we calculate the Kählerian term for a general gauge group. When the gauge group is in the adjoint representation this theory becomes $N = 2$ Yang-Mills theory without hypermultiplets. We show that the one-loop corrections we calculate in $N = 1$ superspace must come from a *non-holomorphic* function of the chiral $N = 2$ superfield W and its conjugate $\bar{W}$.

Our first example is the Wess-Zumino model with $N = 1$ chiral superfield, ϕ, and the most general renormalisable action for this case is :

$$S[\phi, \bar{\phi}] = \int d^8 z \bar{\phi}\phi + \int d^6 z \frac{m\phi^2}{2} + \frac{\lambda\phi^3}{3!} + \text{h.c.}. \tag{2}$$

The effective potential is expressed in terms of the superfield Φ and to begin with we set $m = 0$. The two vertices we have are $\lambda\Phi^3/3!$ and its conjugate $\lambda\bar{\Phi}^3/3!$. In the massless case we have only the $\langle\bar{\Phi}\Phi\rangle$ propagator and consequently the two types of vertex must be placed alternately around the loop, giving $2n$ vertices in each loop, n of each type, as shown in Figure 1.
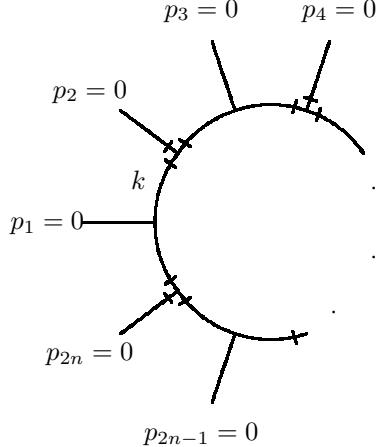

Figure 1: Order n contribution to the one loop effective potential for the massless chiral field.

In superspace the expression for such a graph is

$$\Gamma[n] = \int \frac{d^4x \, d^4k \, d^4\theta_1 \ldots d^4\theta_{2n}}{2n(16k^2)^{2n}} \left\{\lambda\Phi_1 \bar{D}_1^2 D_2^2 \delta_{12}\right\} \lambda\bar{\Phi}_2\delta_{23} \left\{\lambda\Phi_3 \bar{D}_3^2 D_4^2 \delta_{34}\right\} \lambda\bar{\Phi}_4\delta_{45}$$

$$\ldots \left\{\lambda\Phi_{(2n-1)} \bar{D}_{(2n-1)}^2 D_{(2n)}^2 \delta_{(2n-1)(2n)}\right\} \lambda\bar{\Phi}_{(2n)}\delta_{(2n)1} \tag{3}$$

and by rearranging the covariant derivatives we can perform all but two of the θ_i integrations. In Eq. (3) we have included the combinatoric factor, $1/2n$, which is related to the symmetry of the loop [1], in this case n for rotation and n for reflection. Although the calculation in Eq. (3) has

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been done for the massless Wess-Zumino model the same procedure can be used for all the theories considered in this paper, giving the general expression

$$\Gamma[n] = \frac{1}{sn} \int \frac{d^4x \, d^4k \, d^4\theta_1 \, d^4\theta_2}{16\pi^4} \delta_{12} G D^2 \bar{D}^2 (G D^2 \bar{D}^2 (G \ldots D^2 \bar{D}^2 \delta_{12} \ldots)) \tag{4}$$

where we have generalised the combinatoric factor to $1/sn$ where s is an integer determined by the symmetry of the loop. In this paper it is 2 for the Wess-Zumino model and 1 for the gauge theories. In the theories we study in this paper we can combine the vertices into one effective vertex which we have called G. We then connect n copies of G together to form the most general loop. These loops are then summed over n from 1 to ∞ to give Γ. From Eq. (3) we see that in the massless Wess-Zumino model

$$G = \frac{\lambda^2 \bar{\Phi} \Phi}{16r^2}$$

where $r = k^2$.

We can expand out the derivatives in Eq. (4) using Leibniz' rule and then use the well known δ-function rules in superspace to deduce that only terms with equal even multiples of the superspace covariant derivatives acting on the external lines can contribute.

Returning to the massless Wess-Zumino model we see that, since Φ is chiral, we need not consider terms of the form $\bar{D}_{\dot{B}} D_A \Phi$ because they are zero when the external momenta, p_i are zero. This enables us to rearrange derivatives without getting factors dependent on the p_i. We can calculate the Kählerian term, K, by not alowing any of the covariant derivatives to act on the external lines, G, and then summing the graphs for all n. We write $\int d^4k = \int \pi^2 r dr$ and the general expression is

$$K = \int \frac{dr \, d^4\theta}{16s\pi^2} \ln\{1 + 16Gr\}. \tag{5}$$

For the massless Wess-Zumino model this gives

$$K = \int \frac{dr \, d^4\theta}{32\pi^2} \ln\left\{1 + \frac{\lambda^2 \bar{\Phi} \Phi}{r}\right\}. \tag{6}$$

After integrating over r we impose wave-function renormalisation. Coleman and Weinberg [1] give the standard renormalisation conditions for a non-supersymmetric massive scalar field theory and discuss the necessary modifications when the mass in the original Lagrangian is zero. In a massive theory we can impose renormalisation conditions at $\Phi = \Phi_0 = 0$ but in the massless case there is a divergent logarithm at this point and so we must renormalise at $\Phi = \Phi_0 \neq 0$ to avoid this. Hence we demand [2]

$$\frac{\partial^2 \Gamma}{\partial \Phi \partial \bar{\Phi}}\Bigg|_{\substack{\Phi = \Phi_0 \\ \bar{\Phi} = \bar{\Phi}_0}} = 1. \tag{7}$$

which leads to our final result

$$K_{ren} = \int \frac{d^8z}{32\pi^2} \lambda^2 \bar{\Phi} \Phi \left[2 - \ln\left\{\frac{\bar{\Phi} \Phi}{\mu^2}\right\}\right]. \tag{8}$$

In Ref. [2] the same result is achieved, with $\mu^2 = \bar{\Phi}_0 \Phi_0$, using functional and operatorial methods in superspace instead of the more direct approach of supergraphs.

We move on to the calculation of terms in the expansion of Eq. (4) with exactly two D_A and two $\bar{D}_{\dot{B}}$ acting on the external lines, G. As an example of how this could occur in the nth order term let the first '$p-1$' $D^2\bar{D}^2$ operators in Eq. (4) act on the δ-function to give

$$\int \frac{d^4x\ d^4k\ d^4\theta_1\ d^4\theta_2}{16\pi^4 sn}\delta_{21}G\ D^2\bar{D}^2(G\ D^2\bar{D}^2(\ldots D^2\bar{D}^2\ (G^p\left(D^2\bar{D}^2\ \right)^p\delta_{12})\ldots)).$$

By hypothesis the next $D^2\bar{D}^2$ has at least one derivative acting on a G term. One possibility is that *all four* of these derivatives act on the G^p term and so all the remaining $D^2\bar{D}^2$ must act on the $D^2\bar{D}^2\delta$, giving

$$\int \frac{d^4x\ d^4k\ d^4\theta_1\ d^4\theta_2}{16\pi^4 sn}G^{n-p}\ D^2\bar{D}^2(G^p)\,\delta_{21}\left(D^2\bar{D}^2\ \right)^{n-1}(\delta_{12}).$$

where the θ_2 integral is now trivial. We could allow other combinations of covariant derivatives to act on the G^p term and the remaining derivatives from the $D^2\bar{D}^2$ would act on the final δ-function. In this case we would need to get the remainder of our required four derivatives from subsequent $D^2\bar{D}^2$ operators, parts of which act on the external lines, G, and parts of which act on the δ-function by Leibniz rule. In total there are nine ways of getting four derivatives to act on the external lines and they are displayed in Table 1 where $\eta(n)=(-16r)^n/nr$ and the c_i are combinatoric factors needed to combine the results into the final answer. From this table we find, after some algebra,

i	c_i	I_i
1	1	$\sum_{n=2}^{\infty}\eta(n)\sum_{p=1}^{n-1}G^{-p+n}\,D^2\bar{D}^2(G^p)$
2	1	$\sum_{n=3}^{\infty}\eta(n)\sum_{p=1}^{n-2}G^{-1-p+n}\,\bar{D}^2(G\,D^2(G^p))$
3	-2	$\sum_{n=3}^{\infty}\eta(n)\sum_{p=1}^{n-2}G^{n-p-1}\,\bar{D}^{\dot{B}}(G\,D^2\bar{D}_{\dot{B}}(G^p))$
4	-2	$\sum_{n=3}^{\infty}\eta(n)\sum_{p=1}^{n-2}G^{-1-p+n}\,D^A\bar{D}^2(G\,D_A(G^p))$
5	-2	$\sum_{n=4}^{\infty}\eta(n)\sum_{p=1}^{n-3}G^{-2-p+n}\,\bar{D}^{\dot{B}}(G\,D^A\bar{D}_{\dot{B}}(G\,D_A(G^p)))$
6	-2	$\sum_{n=5}^{\infty}\eta(n)\sum_{q=3}^{n-2}\sum_{p=1}^{q-2}G^{n-q-1}\,\bar{D}^{\dot{B}}(G\,D^A(G^{q-p-1}\,\bar{D}_{\dot{B}}(G\,D_A(G^p))))$
7	-2	$\sum_{n=4}^{\infty}\eta(n)\sum_{q=3}^{-1+n}\sum_{p=1}^{q-2}G^{-q+n}\,D^A\bar{D}^{\dot{B}}(G^{-1-p+q}\,\bar{D}_{\dot{B}}(G\,D_A(G^p)))$
8	-2	$\sum_{n=4}^{\infty}\eta(n)\sum_{q=2}^{n-2}\sum_{p=1}^{q-1}G^{-1-q+n}\,\bar{D}^{\dot{B}}(G\,D^A(G^{-p+q}\,D_A\bar{D}_{\dot{B}}(G^p)))$
9	-2	$\sum_{n=3}^{\infty}\eta(n)\sum_{q=2}^{-1+n}\sum_{p=1}^{q-1}G^{-q+n}\,D^A\bar{D}^{\dot{B}}(G^{-p+q}\,D_A\bar{D}_{\dot{B}}(G^p))$

Table 1: Contributions to the four derivative term.

that for a general G

$$\sum_{i=1}^{9}c_iI_i\ =\ \frac{-\left(D^AG\bar{D}^{\dot{B}}G-(1+G)D^A\bar{D}^{\dot{B}}G\right)\left(D_AG\bar{D}_{\dot{B}}G-(1+G)D_A\bar{D}_{\dot{B}}G\right)}{2r(1+G)^4} \tag{9}$$

In the massless Wess-Zumino model G corresponds to $\lambda^2\bar{\Phi}\Phi/(16r^2)$ and after simplification and integration over r we find

$$F_1\ =\ \int\frac{dr\,d^8z}{(16\pi)^2 s}\sum_{i=1}^{9}c_iI_i\ =\ \int\frac{d^8z}{(4\pi)^2}\frac{D^A\Phi D_A\Phi\bar{D}^{\dot{B}}\bar{\Phi}\bar{D}_{\dot{B}}\bar{\Phi}}{384(\bar{\Phi}\Phi)^2}. \tag{10}$$

The form of this result was given in Ref. [2] but the coefficient is given only in the form of an integral, for which the result is not known. Our technique gives the coefficient explicitly and the

complete result for the massless Wess-Zumino model is

$$\Gamma = \int \frac{d^8z}{(4\pi)^2} \frac{\lambda^2\bar{\Phi}\Phi}{2} \left[2 - \ln\left\{\frac{\bar{\Phi}\Phi}{\mu^2}\right\}\right] + \frac{(D\Phi)^2(\bar{D}\bar{\Phi})^2}{384(\bar{\Phi}\Phi)^2} + O(D^4, \bar{D}^4) \tag{11}$$

with $(D\Phi)^2(\bar{D}\bar{\Phi})^2 = D^A\Phi D_A\Phi \bar{D}^{\dot{B}}\bar{\Phi}\bar{D}_{\dot{B}}\bar{\Phi}$.

When we consider the massive Wess-Zumino model we can no longer neglect graphs containing adjacent identical vertices as we now have $\langle\Phi\Phi\rangle$ and $\langle\bar{\Phi}\bar{\Phi}\rangle$ propagators aswell as $\langle\bar{\Phi}\Phi\rangle$. To account for this we replace each vertex in the massless diagrams by $q+1$ adjacent copies of this vertex and sum q from 0 to ∞ at each point. This corresponds to defining a dressed vertex so that the chiral vertex becomes

$$\lambda\Phi\bar{D}^2 \sum_{q=0}^{\infty} \left(\frac{-m\lambda\Phi}{k^2+m^2}\right)^q = \frac{\lambda\Phi}{1 + \frac{m\lambda\Phi}{k^2+m^2}}\bar{D}^2 \tag{12}$$

and by using it we allow for all possible numbers of adjacent chiral vertices from one to infinity between any two neighbouring anti-chiral vertices. The resulting contribution behaves just like a single chiral vertex as far as the D-algebra is concerned and after similarly dressing the anti-chiral vertex we can only connect the dressed vertices using the $\langle\bar{\Phi}\Phi\rangle$ propagator as we have allowed for all possible occurrences of the other two propagators. What is more, the symmetry arguments for the combinatoric factors will follow through and once again $s = 2$. The situation is very similar to the massless case but we have a new expression for the external lines, G given by

$$G = \frac{\lambda^2\bar{\Phi}\Phi}{16(r + m\Psi)(r + m\bar{\Psi})}$$

where $\Psi = \lambda\Phi + m$. After integrating and renormalising using Eq. (7), we get

$$K_{ren} = \int \frac{d^8z}{32\pi^2} \bar{\Psi}\Psi \left[2 - \ln\left(\frac{\bar{\Psi}\Psi}{\mu^2}\right)\right]. \tag{13}$$

The same arguments about dressing vertices give

$$F_1 = \int \frac{d^8z}{(4\pi)^2} \frac{D^A\Psi D_A\Psi \bar{D}^{\dot{B}}\bar{\Psi}\bar{D}_{\dot{B}}\bar{\Psi}}{384(\bar{\Psi}\Psi)^2} \tag{14}$$

for the massive non-Kählerian term. The complete result is thus

$$\Gamma = \int \frac{d^8z}{(4\pi)^2} \frac{\bar{\Psi}\Psi}{2} \left[2 - \ln\left(\frac{\bar{\Psi}\Psi}{\mu^2}\right)\right] + \frac{(D\Psi)^2(\bar{D}\bar{\Psi})^2}{384(\bar{\Psi}\Psi)^2} + O(D^4, \bar{D}^4) \tag{15}$$

These results agree with those found by the method of Ref. [2] if $\mu^2 = \bar{\Psi}_0\Psi_0$ and we take the value of the unknown coefficient to have the value given.

We move on to the case of N = 1 U(1) gauge theory. The action is

$$\frac{1}{64g^2} \int \mathrm{d}^6z \, W^A W_A + \int \mathrm{d}^8z \, \bar{\Phi}e^{gV}\Phi + \text{gauge fixing} + \text{ghosts} \tag{16}$$

where $W_A = \bar{D}^2 e^{-gV} e^{gV} D_A V$ so that for a U(1) gauge group $W_A = g\bar{D}^2 D_A V$. We have two propagators, $\langle\bar{\Phi}\Phi\rangle$ and $\langle VV\rangle$ and the interactions come from the $\bar{\Phi}e^{gV}\Phi$ term. They give vertices with any number of V lines and one $\bar{\Phi}$ and one Φ line, as indicated in Figure 2.
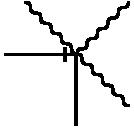
Figure 2: The general form of an interaction vertex from the $\bar{\Phi}e^{gV}\Phi$ term in N = 1 U(1) gauge theory.

Exactly two of these lines must be internal giving the options shown in Figure 3 where horizontal lines are internal, the V lines are the wiggly lines and although we have only shown only one external V line in each case there could infact be any number, from zero to infinity, for each graph. However Figure 3(d) requires at least one external V line to be non-zero.
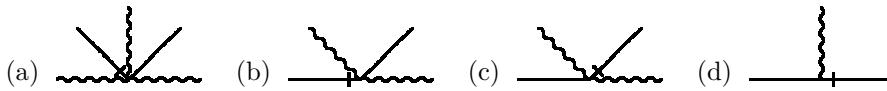

Figure 3: Possible ways of orienting vertices in a loop in N = 1 U(1) gauge theory.

As we have seen the key to the solution is to find a general expression for all the allowed loops and cast it in the form of Eq. (4). To do this we observe that the vertex in Figure 3(b) cannot use the $\langle VV \rangle$ propagator to join to another identical vertex in the Kählerian approximation since $D^2 D^2 = 0$. Similarly for Figure 3(c) and so any loop with Figure 3(b) in has an equivalent number of Figure 3(c) vertices. In fact they go in pairs joined by their $\langle VV \rangle$ propagator but this propagator can be dressed with any number of copies of Figure 3(a) as shown in Figure 4.

$$\sum_{n=0}^{\infty} \quad \underline{\quad\quad} \left(\quad \bigvee \quad \right)^n \underline{\quad\quad} \quad = \quad \underline{\quad\quad\quad} \tag{17}$$

Figure 4: The dressed gauge propagator.

We could form a valid loop by connecting n copies of this vertex together, but because Figure 3(d) contributes the same internal lines to the loop, it is equally valid to form a new loop by substituting Figure 4 with Figure 3(d) at any point in the loop . This corresponds to creating a combined vertex by adding these two together as in Figure 5.   Joining n such a vertices together and performing the $\sum_{n=1}^{\infty}$ gives all contributions to the effective superpotential in the form of Eq. (4). By this procedure we have neglected only those graphs which consist of Figure 3(a). These are identically zero because they contain no $D^2 \bar{D}^2$ between the δ-functions.

Using the super-Feynman rules and Dyson's formula for the combinatoric factor of a tree diagram we find that

$$G \quad = \quad \frac{1}{16r} \left\{ e^{gV} - 1 + \frac{\beta \bar{\Phi} e^{2gV} \Phi}{1 - \beta \bar{\Phi} e^{gV} \Phi} \right\}, \tag{18}$$

with $\beta = \frac{-g^2}{r}$. There is no reflection symmetry in this case so $s = 1$ in Eq. (4) and we get

$$K \quad = \quad \int \frac{dr \, d^8 z}{16\pi^2} \left\{ gV - \ln \left(1 + \frac{g^2 \bar{\Phi} e^{gV} \Phi}{r} \right) \right\}. \tag{19}$$

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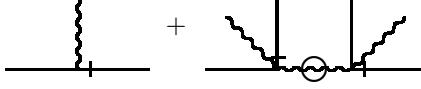
Figure 5: The combined vertex from which all one loop graphs can constructed in N = 1 U(1) gauge theory.

We can ignore the gV term as it would not appear in a dimensional reduction scheme [4]. We integrate and then renormalise using Eq. (7) where $\Phi_0 \neq 0$ because Φ is massless and we must set $V = 0$ at the renormalisation point. We get

$$K = -\int \frac{d^8 z}{(4\pi)^2} g^2 \bar{\Phi} e^{gV} \Phi \left\{ 2 - \ln\left(\frac{\bar{\Phi} e^{gV} \Phi}{\mu^2}\right) \right\}. \tag{20}$$

where $\mu^2 = \bar{\Phi}_0 \Phi_0$.
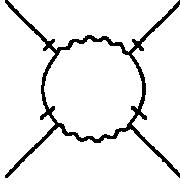
Figure 6: Loop containing adjacent D^2 which can only contribute in the non-Kählerian case.

We can use Eq. (18) to calculate some contributions to the non-Kählerian term aswell, but because we are no longer in the Kählerian approximation we cannot simply ignore graphs with adjacent D^2 operators as we did in the Kählerian case. Hence we must also include the graph shown in Figure 6. In order that this is non-zero one of the adjacent D^2 must act on the external lines, or else we get $D^2 D^2$ which is zero. Similarly for the adjacent $\bar{D}^2$, and thus we immediately get the required number of external covariant derivatives. It remains to dress the propagators with all allowed vertices, without letting any derivatives act externally. For a U(1) gauge group the only terms we find are infrared divergent. It can be shown that in a full treatment with massive scalars these terms have appropriate factors of $1/m^2$ associated with them. Hence the only contributions to the non-Kählerian term come from substituting the expression for G into Eq. (9) as before.

We are interested in two terms in particular: the first where none of the derivatives act on external V lines, the second where they all do and we only include terms of $O(V^2)$. In the former case we get

$$F_1 = \int \frac{d^8 z}{(4\pi)^2} \frac{D^A \Phi D_A \Phi \bar{D}^{\dot{B}} \bar{\Phi} \bar{D}_{\dot{B}} \bar{\Phi}}{192 (\bar{\Phi}\Phi)^2} \tag{21}$$

whilst in the latter we get

$$F_2 = \frac{1}{16g^2} \int \frac{d^8 z}{(4\pi)^2} \frac{g^2 D^A \bar{D}^{\dot{B}} V D_A \bar{D}_{\dot{B}} V}{2} \left\{ \ln\left(\frac{\bar{\Phi}\Phi}{\mu^2}\right) \right\}. \tag{22}$$

This is clearly supersymmetric but we can use $\int d^2\bar{\theta} = -\bar{D}^2/4$ to write this in terms of $W^A W_A$. In order to show that Eq. (22) is gauge invariant we must use the fact that the external momenta

are zero. Finally, in order to renormalise, we have set the coefficient of $W^A W_A$ to be $1/64g^2$ at $\Phi = \Phi_0$. The full result is thus

$$\Gamma = \int \frac{d^4\theta}{(4\pi)^2} g^2 \bar{\Phi} e^{gV} \Phi \left\{ \ln\left(\frac{\bar{\Phi}\Phi}{\bar{\Phi}_0\Phi_0}\right) - 2 \right\} + \frac{D^A \Phi D_A \Phi \bar{D}^{\dot{B}} \bar{\Phi} \bar{D}_{\dot{B}} \bar{\Phi}}{192(\bar{\Phi}\Phi)^2} - $$
$$\frac{1}{64g^2} \int \frac{d^2\theta}{(4\pi)^2} \frac{g^2 W^A W_A}{2} \left\{ \ln\left(\frac{\bar{\Phi}\Phi}{\mu^2}\right) \right\} + \ldots + O(D^4, \bar{D}^4) \tag{23}$$

where the $\ldots$ denote terms of $O(D^2, \bar{D}^2)$ which do not interest us.
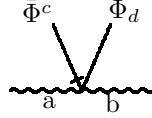
Figure 7: The non-Abelian vertex used to dress the gauge propagator.

We can extend our U(1) arguments to the case of Yang-Mills theory with a general gauge group, but we must obtain a new expression for the combined vertex in Figure 5. For simplicity we shall set all external V lines to zero. The vertex shown in Figure 3(a) becomes that shown in Figure 7. Hence we define

$$A^{ab} = \frac{-g^2}{2} \bar{\Phi} \left\{ T^a, T^b \right\} \Phi. \tag{24}$$

In the absence of external lines the combined vertex becomes

 $= -g^2 (T^c)_a{}^e \Phi_e \left(\frac{1}{r-A}\right)^{cd} \bar{\Phi}^f (T^d)_f{}^b \left(\frac{\bar{D}^2 D^2}{16r}\right). \tag{25}$

Hence

$$G = -g^2 (T^c)_a{}^e \Phi_e \left(\frac{1}{16r(r-A)}\right)^{cd} \bar{\Phi}^f (T^d)_f{}^b$$

and

$$K = \int \frac{dr\, d^8 z}{16\pi^2} \left[\text{Tr} \ln (r - F) - \text{Tr} \ln (r - A) \right] \tag{26}$$

where

$$F = \frac{g^2}{2} \bar{\Phi} \left[T^a, T^b \right] \Phi. \tag{27}$$

We can perform the integral and by extending Eq. (7) to matrix form we eventually get

$$K = \text{Tr} \int \frac{d^8 z}{16\pi^2} \left[\Delta - \frac{\bar{\Phi}\Phi}{\bar{\Phi}_0\Phi_0} \bar{\Phi}_0^a \frac{\partial^2 \Delta}{\partial \bar{\Phi}^a \partial \Phi_b} \bigg|_{\substack{\Phi=\Phi_0 \\ \bar{\Phi}=\bar{\Phi}_0}} \Phi_{0b} \right]. \tag{28}$$

where $\Delta = F \ln |F| - A \ln |A|$, i.e.

$$
\begin{aligned}
\Delta^{ac} &= \frac{g^2}{2} \bar{\Phi} \left[T^a, T^b \right] \Phi \left(\ln \left| \frac{g^2}{2} \bar{\Phi} [T,T] \Phi \right| \right)^{bc} \\
&\quad + \frac{g^2}{2} \bar{\Phi} \left\{ T^a, T^b \right\} \Phi \left(\ln \left| \frac{g^2}{2} \bar{\Phi} \{T,T\} \Phi \right| \right)^{bc}.
\end{aligned}
\tag{29}
$$

Because all the matrices we have to deal with are linear in each of $\bar{\Phi}$ and Φ we can infact simplify this expression further, giving the general result

$$
K = \int \frac{d^8 z}{16\pi^2} \left(\alpha_0 \bar{\Phi} \Phi + \mathrm{Tr}\Delta \right).
\tag{30}
$$

where

$$
\alpha_0 = -g^2 C(R) - \frac{\Delta_0}{\bar{\Phi}_0 \Phi_0}
\tag{31}
$$

and

$$
(T^s)_a{}^b (T^s)_b{}^c = \delta_a{}^b C(R).
$$

We have investigated the form of the $W^A W_A$ term in the non-Abelian case and we shall publish our results in a forthcoming paper.

We now evaluate Eq. (30) when the gauge group is SU(2) in the adjoint representation. In this case the theory possesses $N = 2$ supersymmetry and

$$
(T^a)_b{}^c = \epsilon_{abc}
$$

the totally anti-symmetric tensor in three dimensions. We can diagonalise the matrices in Eq. (30) and the trace becomes a sum over *non-zero* eigenvalues. To prove this we must diagonalise before we integrate over r. For A the eigenvalues are

$$
-a, \; -d_+, \; -d_- \quad \text{where} \quad d_\pm = \frac{-g^2}{2} (\bar{\Phi}\Phi \pm \sqrt{\bar{\Phi}^2 \Phi^2})
$$

and for F they are

$$
0, \; \pm f \quad \text{where} \quad f = \frac{g^2}{2} \sqrt{(\bar{\Phi}\Phi)^2 - \bar{\Phi}^2 \Phi^2}.
$$

Hence

$$
K = \int \frac{d^4\theta}{16\pi^2} \alpha_0 \bar{\Phi}\Phi + 2a \ln a + d_+ \ln d_+ + d_- \ln d_-.
\tag{32}
$$

In this paper we have calculated our results for $N = 1$ supersymmetric theories. Yang-Mills theory also possesses $N = 2$ supersymmetry when the matter fields are in the adjoint representation of the gauge group. For the $N = 1$ action we have considered the resulting theory is $N = 2$ Yang-Mills theory without hypermultiplets. By combining the two $N = 1$ fields Φ and W_A into a single chiral $N = 2$ superfield W we can write the action in $N = 2$ superspace. It was shown in Ref. [5] that a holomorphic function $\mathcal{F}$ of the $N = 2$ Yang-Mills chiral superfield W can be written in $N = 1$ language as

$$
\int d^4\theta \, \mathrm{Im} \left(\bar{\Phi} \frac{\partial \mathcal{F}}{\partial \Phi} \right) + \frac{1}{2} \int d^2\theta \, W^A W_A \mathrm{Im} \left(\frac{\partial^2 \mathcal{F}}{\partial \Phi^2} \right).
\tag{33}
$$

Thus if there are holomorphic corrections to the one loop effective potential in $N = 2$ Yang-Mills theory the $N = 1$ Kählerian term should be related to the holomorphic function $\mathcal{F}$ as above. In order for this to be so we need to use a relation analagous to

$$\ln \bar{\Phi}\Phi = \ln \bar{\Phi} + \ln \Phi,$$

enabling us to split the logarithm in Eq. (28) into a sum of chiral and anti-chiral parts. There is no such relation for $\ln \bar{\Phi}\{T^a, T^b\}\Phi$ or $\ln \bar{\Phi}\left[T^a, T^b\right]\Phi$ and so these terms must come from a *non-holomorphic* function of W and $\bar{W}$.

In this paper we have applied the supergraph technique to the Wess-Zumino model to calculate both the Kählerian contribution and the first non-Kählerian contribution to the one loop effective superpotential. We have extended these methods to N = 1 U(1) gauge theory where we use our non-Kählerian calculations to find not only the term in $D^A\Phi D_A\Phi \bar{D}^{\dot{B}}\bar{\Phi}\bar{D}_{\dot{B}}\bar{\Phi}$ but also the $W^A W_A$ term. In addition we have also calculated the Kählerian term in Yang-Mills theory for a general gauge group and and we conclude that it can only be generated from a non-holomorphic correction to the effective potential in $N = 2$ Yang-Mills theory.

Note Added: After we had finished this work we became aware of a paper [6] in which the authors calculate the one loop Kählerian correction to effective super-potential for $N = 2$ Yang-Mills Theory. Like us they find that there are non-holomorphic corrections to the $N = 2$ Yang-Mills effective potential and whilst they do not calculate any non-Kählerian terms they do calculate the function of the $N = 2$ fields which gives rise to these non-holomorphic corrections.

Note Added In Proof : In the above we have found the Kählerian contribution to the effective superpotential for Yang-Mills theory in the absence of cubic interaction terms. In this note we extend our result slightly by adding a term of the form

$$\int d^2\theta \frac{1}{3!}\lambda^{abc}\Phi_a\Phi_b\Phi_c + \text{ h.c.}$$

to the action of Eq. (16), and we obtain the one loop Kählerian contribution to the superpotential for the most general renormalisable $N = 1$ supersymmetric theory. We find the Kählerian contribution to the superpotential by considering all graphs of the form shown in Figure 1 and then dressing the $\langle\Phi\bar{\Phi}\rangle$ propagator with the vertex shown in Eq. (25). Similarly the $\langle\bar{\Phi}\Phi\rangle$ is dressed with the transpose of this vertex. We use Eq. (25) to define C as

$$_a \underline{\quad\quad\quad\quad}_b \;=\; -g^2(T^c)_a{}^e \Phi_e \left(\frac{1}{r - A}\right)^{cd} \bar{\Phi}^f (T^d)_f{}^b \left(\frac{\bar{D}^2 D^2}{16r}\right) \;=\; C_a{}^b \left(\frac{\bar{D}^2 D^2}{16r}\right) \quad (34)$$

The dressed matter propagators are then given by

$$\langle\Phi\bar{\Phi}\rangle_{\text{dressed}} \;=\; \underline{\quad\quad} \;=\; \frac{1}{r}\sum_{n=0}^{\infty}(-C)^n \;=\; \frac{1}{r}\frac{1}{1+C} \quad (35)$$

$$\langle\bar{\Phi}\Phi\rangle_{\text{dressed}} \;=\; \underline{\quad\quad} \;=\; \frac{1}{r}\sum_{n=0}^{\infty}(-C^T)^n \;=\; \frac{1}{r}\frac{1}{1+C^T} \quad (36)$$

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We can generate all possible graphs containing at least one $\lambda\Phi^3/3!$ vertex, by forming a loop with n of the vertices shown in Eq. (37).

$$c \underset{\quad}{\underline{\hspace{2cm}}}_{g} \;=\; G^c{}_g \bar{D}^2 D^2 \;=\; (\lambda^{abc}\Phi_a)\left(\frac{1}{1+C}\right)^d_b (\lambda_{edf}\bar{\Phi}^e)\left(\frac{1}{1+C^T}\right)^f_g \left(\frac{\bar{D}^2 D^2}{16r^2}\right) \quad (37)$$

Substituting this expression for G into Eq. (5) gives the corresponding contribution to the super-potential.

In addition to this we must include the contribution from all loops without any $\lambda\Phi^3/3!$ vertices, but this is simply the result obtained in Eq. (26) where $\lambda^{abc} = 0$. Rewriting this in terms of C and adding these two contributions together gives the full massless result

$$K \;=\; \int \frac{d^8z\,dr}{32\pi^2}\left[\mathrm{Tr}\ln\left(1+\frac{1}{r}P\left(\frac{1}{1+C}\right)\bar{P}\left(\frac{1}{1+C^T}\right)\right)+2\,\mathrm{Tr}\ln\left(1+C\right)\right] \quad (38)$$

where $P^{bc} = \lambda^{abc}\Phi_a$.

We can now deduce the result for the massive theory by treating the mass term, $m^{ab}\Phi_a\Phi_b/2!$, as an interaction vertex. This vertex behaves in exactly the same way as the $\lambda\Phi^3/3!$ vertex, but with P^{bc} replaced by m^{bc}. We can thus simply add these vertices together, and the massive results then follow from the massless results under the transformation

$$P^{bc} \;\rightarrow\; \Psi^{bc} \;=\; \lambda^{abc}\Phi_a + m^{bc}, \quad (39)$$

as we found explicitly for the Wess-Zumino model.

We get

$$K \;=\; \int \frac{d^8z\,dr}{32\pi^2}\left[\mathrm{Tr}\ln\left(1+\frac{1}{r}\Psi\left(\frac{1}{1+C}\right)\bar{\Psi}\left(\frac{1}{1+C^T}\right)\right)+2\,\mathrm{Tr}\ln\left(1+C\right)\right] \quad (40)$$

which is the Kählerian contribution for the most general massive $N = 1$ renormalisable theory with cubic interactions and a general gauge group.

After we had obtained these results we became aware of a recent paper, which appeared on the hep-th archive before this 'note added in proof' was written, and which derived the results by a different method, for a general gauge choice. We have derived them for the super Fermi-Feynman gauge, $\xi = 1$ in their notation, as explained along with our conventions in Ref. [3]. Although the two expressions look different, after some algebra we find that they are the same, provided that the matrix

$$\left[\frac{1}{k^2 - (S+T)}\right]$$

in Equation (4.17) of Ref. [7] is replaced by its transpose.

In this paper [7] the authors also find an additional imaginary contribution to the Kählerian term for SU(2) Yang-Mills theory. This term can be recovered from our results by removing the modulus signs in Eq. (29). However, they also note that such a term does not contribute to the physics.

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